B.Sc. Part II Paper - III

ABSTRACT ALGEBRA

GROUP | continued

1. Verify whether, the set of integers with respect to binary operation "multiplication" is a group or not.

Solution Here, $I = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$

(a) Closure : $(-3) \times (-7) = 21$ and $-3, -7, 21 \in I$

$4 \times (-15) = -60$ where $4, -15, -60 \in I$

closure axiom satisfied.

(c) Identity : Here, 1 is the identity ~~axiom.~~

$(-4) \times 1 = -4$

$23 \times 1 = 23$

identity axiom satisfied

(b) Associative

$3, 2, 5 \in I$ and

$3 \times (2 \times 5) = (3 \times 2) \times 5$

$16 \times \{(-4) \times 3\} = [16 \times (-4)] \times 3$

Associative axiom satisfied.

(d) Inverse $\quad 2 \in I$

but there is no other element $\in I$ s.t

$2 \times a = 1 \Rightarrow a = \frac{1}{2}$ but $\frac{1}{2} \notin I$

$\therefore$ Inverse axiom not satisfied.

Hence $(I, \times)$ is not a group.

Q. The set of rational numbers is a group w.r.to binary operation 'addition'.

Soln Here $\mathbb{Q}$ = set of rational numbers

$$= \left\{ \frac{p}{q} : p, q \in I, q \neq 0 \right\}$$

(a) Closure : Sum of two rational nos. is again a rational no.

ex : $\frac{3}{4}, -\frac{7}{12} \in \mathbb{Q}$, ~~$\frac{3}{4}$~~

and their sum $= \frac{3}{4} + \left(-\frac{7}{12}\right) = \frac{1}{6} \in \mathbb{Q}$.

(b) Associative

Take ~~to~~ $\frac{2}{3}, \frac{5}{7}, \frac{-4}{9}$. These are elements of $\mathbb{Q}$.

Now $\frac{2}{3} + \left[\frac{5}{7} + \left(-\frac{4}{9}\right)\right]$ can be verified easily as equal to $\left[\left(\frac{2}{3} + \frac{5}{7}\right) + \left(-\frac{4}{9}\right)\right]$

(c) Identity

Take $-\frac{7}{15} (\in \mathbb{Q})$.

$$\frac{-7}{15} + 0 = 0 + \frac{-7}{15} = -\frac{7}{15}.$$

is the identity element. and $0 \in \mathbb{Q} \Rightarrow 0$ is the identity element.

(d) Inverse Take $\frac{-17}{35}$. $\frac{-17}{35} \in \mathbb{Q}$.

$$\left(-\frac{17}{35}\right) + a = 0 = a + \left(-\frac{17}{35}\right).$$

Obviously, $a = \frac{17}{35}$ and $\frac{17}{35} \in \mathbb{Q}$.

So inverse axiom is satisfied.

Hence $(\mathbb{Q}, +)$ is a group.

9. The set of non-zero rational numbers is a group w.r.to multiplication.

Proof

Let $Q_1 = Q - \{0\}$

(a) Closure

closure law satisfied

Take $\frac{7}{3}, \frac{8}{15}$. $\frac{7}{3}, \frac{8}{15} \in Q_1$

$\frac{7}{3} \times \frac{8}{15} = \frac{56}{45}$ and $\frac{56}{45} \in Q_1$

Take $\frac{1}{2}, \frac{-3}{7}$. $\frac{1}{2}, \frac{-3}{7} \in Q$

Now, $\frac{1}{2} \times \frac{-3}{7} = -\frac{3}{14}$ and $-\frac{3}{14} \in Q_1$.

(b) Associative

Asso. law satisfied

Students → do it yourself.

Take any 3-non-zero-rational nos. and verify that $a*(b*c) = (a*b)*c$

(c) Identity

Identity exists

1 is the identity element.

Take any rational no. Take $\frac{17}{38}$.

$\frac{17}{38} \times 1 = \frac{17}{38} = 1 \times \frac{17}{38}$.

(d) Inverse $\frac{2}{3} \in Q_1$.

Inverse exists

Now $\frac{2}{3} \times a = 1 = a \times \frac{2}{3}$

$\therefore a = \frac{3}{2}$ and $\frac{3}{2} \in Q_1$

Hence Q_1 is a group w.r.to $(\times)$.

[Q] If a, b be any two elements of a group (G, o) then prove that

$(aob)^{-1} = b^{-1} o a^{-1}$.

Proof Here, (G, o) is a group.

Let 'e' be identity element of G.

Since $a, b \in G$

$\Rightarrow aob \in G$, [by closure axiom]

we know that, by inverse axiom that

if $a, b \in G$ ~~then~~ and $ab = e = ba$

$\Rightarrow a^{-1} = b$ and $b^{-1} = a$.

Take the expression

$(aob) o (b^{-1} o a^{-1})$.

$= [a o (b o b^{-1})] o a^{-1}$ [by associative law]

$= [a o e] o a^{-1}$ [by inverse law, $a o a^{-1} = e$]

$= a o a^{-1}$ [by identity law, $a o e = a$]

$= e$ [by inverse axiom]

$\therefore (aob) o (b^{-1} o a^{-1}) = e$

$\Rightarrow$ Inverse of $aob = b^{-1} o a^{-1}$

$\Rightarrow (aob)^{-1} = b^{-1} o a^{-1}$.